

Stabilization of the spatial oscillations of an elastic system model

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Abstract

A system of partial differential equations describing the spatial oscillations of an Euler-Bernoulli beam with a tip mass is considered. The linear system considered is actuated by two independent controls and separated into a pair of differential equations in a Hilbert space. A feedback control ensuring strong stability of the equilibrium in the sense of Lyapunov is proposed. The proof of the main result is based on the theory of strongly continuous semigroups.

1 Introduction

Dynamical models of flexible-link robot manipulators are generally described by a set of coupled ordinary and partial differential equations, that gives rise to series of mathematical control problems in infinite dimensional spaces [3, 4, 5, 7, 12]. However, finite dimensional approximate models obtained by the assumed modes and finite elements methods are used more frequently for solving the motion planning and stabilization problems [1, 11]. It should be emphasized that the majority of publications in this area is concentrated on planar manipulator models with a free end. To study spatial manipulators with a tip mass, the mathematical model that describes the motion of a multi-link manipulator under the action of gravity and controls (torques and forces) was proposed in [13].

The goal of this paper is to study the stabilization problem of the control system derived in [13] for the particular case of a manipulator with one flexible link.

2 Equations of Motion

A mechanical system consisting of n Euler-Bernoulli beams and a rigid body as a load was introduced in [13]. In this paper, we assume that $n = 1$ and neglect controlled rotations of the load ($\varphi_J = 0$ and $c = 0$ in the notations of [13]). Thus, the beam deflection at time t is defined by functions $y(x, t)$ and $z(x, t)$ in a rotating Cartesian frame, where $x \in [0, l]$ is the spatial coordinate, l is the length of the beam. The above Cartesian frame is obtained from the fixed one by subsequent rotations on the angle $\varphi_T(t)$ (turning angle) and $\varphi_R(t)$ (raising angle). The system is controlled by torques M_T and M_R applied at the bottom part of the beam. For each constant value φ_R^0 , there is the control torque $M_R = M_R^0$ implementing an equilibrium $\varphi_T(t) = 0$, $\varphi_R(t) = \varphi_R^0$, $y(x, t) = 0$, and $z(x, t) = z_0(x)$. The linearized system of differential equation describing oscillations around the equilibrium can be written as follows (see [13]):

$$\ddot{y}(x, t) + \frac{1}{\rho} \left(c_z y''(x, t) \right)'' = \psi_T(x) \ddot{\varphi}_T, \quad x \in (0, l), \quad (1)$$

$$\ddot{z}(x, t) + \frac{1}{\rho} \left(c_y \ddot{z}''(x, t) \right)'' = g \ddot{\varphi}_R \sin \varphi_R^0 - x \ddot{\varphi}_R, \quad (2)$$

$$y|_{x=0} = \ddot{z}|_{x=0} = 0, \quad y'|_{x=0} = \ddot{z}'|_{x=0} = 0, \quad (3)$$

$$\frac{1}{m} (c_z y'')' - \ddot{y} + \psi_T(x) \ddot{\varphi}_T \Big|_{x=l} = 0, \quad (4)$$

$$-c_z y'' - J_3 \ddot{y}' + J_3 \psi_T'(x) \ddot{\varphi}_T \Big|_{x=l} = 0, \quad (5)$$

$$\frac{1}{m} (c_y \ddot{z}'')' + g \ddot{\varphi}_R \sin \varphi_R^0 - \ddot{z} - l \ddot{\varphi}_R \Big|_{x=l} = 0, \quad (6)$$

$$c_y \ddot{z}'' + J_2 (\ddot{\varphi}_R + \ddot{z}') \Big|_{x=l} = 0, \quad (7)$$

$$\begin{aligned} & \left\{ I_0 + (I_1 + J_1) \sin^2 \varphi_R^0 + (I_3 + J_3) \cos^2 \varphi_R^0 + m_0 (R - d \cos \varphi_R^0)^2 + m (R - l \cos \varphi_R^0)^2 + \right. \\ & \quad \left. + \int_0^l (R - x \cos \varphi_R^0)^2 \rho dx \right\} \ddot{\varphi}_T + \int_0^l (R - x \cos \varphi_R^0) \ddot{y} \rho dx + \\ & \quad + \left\{ m R \ddot{y} - (m l \ddot{y} + J_3 \ddot{y}') \cos \varphi_R^0 \right\} \Big|_{x=l} = M_T, \end{aligned} \quad (8)$$

$$\left\{ I_2 + J_2 + m_0 d^2 + m l^2 + \int_0^l x^2 \rho dx \right\} \ddot{\varphi}_R + \int_0^l \ddot{z} x \rho dx + \left\{ m l \ddot{z} + J_2 \ddot{z}' \right\} \Big|_{x=l} -$$

$$\begin{aligned}
& -g \left\{ \int_0^l \tilde{z} \rho dx + m \tilde{z}|_{x=l} + \left(m_0 d + ml + \int_0^l x \rho dx \right) \tilde{\varphi}_R \right\} \sin \varphi_R^0 - \\
& - g \left\{ \int_0^l z_0 \rho dx + m z_0(l) \right\} \tilde{\varphi}_R \cos \varphi_R^0 = \tilde{M}_R,
\end{aligned} \tag{9}$$

where $\tilde{z}(x, t) = z(x, t) - z_0(x)$, $\tilde{\varphi}_R(t) = \varphi_R(t) - \varphi_R^0$, $\tilde{M}_R = M_R - M_R^0$, and

$$\psi_T(x) = x \cos \varphi_R^0 - z_0(x) \sin \varphi_R^0 - R.$$

We use dots to denote derivatives with respect to time t , and primes to denote derivatives with respect to the space variable x . The procedure for computing $z_0(x)$ and M_R^0 is given in [13].

The parameters in (1)-(9) have the following physical meaning: $\rho(x)$ is the mass per unit length of the beam, $c_z(x) = E(x)I_z(x)$, $c_y(x) = E(x)I_y(x)$, $E(x)$ is Young's modulus, $I_z(x)$ and $I_y(x)$ are moments of inertia of the cross section of the beam with respect to the axes z and y , m is the payload mass, J_1 , J_2 , and J_3 are central moments of inertia of the payload, R is the platform radius, I_0 is the moment of inertia of the platform, I_1 , I_2 , and I_3 are moments of inertia of the hub, m_0 is the hub mass, d is the distance between the origin of the rotating Cartesian frame and the hub center of mass.

To simplify these equations we substitute expressions (1), (2), (4)-(7) for $\ddot{y}(x, t)$, $\ddot{z}(x, t)$, $\ddot{y}, \ddot{y}', \ddot{z}, \ddot{z}'|_{x=l}$ into (8), (9) and perform integration by parts with regard for the boundary conditions (3). As a result, equations (8) and (9) take the following form:

$$\ddot{\varphi}_T = u_T, \quad \ddot{\varphi}_R = u_R, \tag{10}$$

where

$$\begin{aligned}
u_T = & \{ I_0 + (I_1 + J_1) \sin^2 \varphi_R^0 + m_0 (R - d \cos \varphi_R^0)^2 + \\
& + (I_3 \cos \varphi_R^0 + J_3 z_0'(l) \sin \varphi_R^0) \cos \varphi_R^0 + \left(m(l \cos \varphi_R^0 - R) z_0(l) + \right. \\
& \left. + \int_0^l (x \cos \varphi_R^0 - R) z_0 \rho dx \right) \sin \varphi_R^0 \}^{-1} \times \{ M_T - (R(c_z y'')' + c_z y'' \cos \varphi_R^0)|_{x=0} \},
\end{aligned} \tag{11}$$

$$\begin{aligned}
u_R = & \{ I_2 + m_0 d^2 \}^{-1} \times \left\{ \tilde{M}_R + c_y \tilde{z}''|_{x=0} + g \left(\int_0^l \tilde{z} \rho dx + m \tilde{z}|_{x=l} + m_0 d \right) \sin \varphi_R^0 + \right. \\
& \left. + g \left(\int_0^l z_0 \rho dx + m z_0(l) \right) \tilde{\varphi}_R \cos \varphi_R^0 \right\}.
\end{aligned} \tag{12}$$

For each $\tilde{\varphi}_R(t)$, $y(\cdot, t)$, $\tilde{z}(\cdot, t)$, formulae (11) and (12) establish a one-to-one correspondence between the torques (M_T, \tilde{M}_R) and angular accelerations (u_T, u_R) . Thus, we may consider $(u_T, u_R) \in \mathbb{R}^2$ as a new control for the linear system (1)-(7), (10).

3 Main Results

Consider the following linear space

$$X = \left\{ \begin{pmatrix} \eta(\cdot) \\ \zeta(\cdot) \\ \phi \\ \omega \\ p \\ q \end{pmatrix} : \begin{array}{l} \eta \in H^2(0, l), \zeta \in L_2(0, l), \\ \eta(0) = \eta'(0) = 0, \\ \phi, \omega, p, q \in \mathbb{R} \end{array} \right\},$$

where $H^k(0, l)$ is the Sobolev space of all functions whose generalized derivatives of order $j = 0, 1, \dots, k$ exist and belong to $L_2(0, l)$. For

$$\xi_1 = \begin{pmatrix} \eta_1 \\ \zeta_1 \\ \phi_1 \\ \omega_1 \\ p_1 \\ q_1 \end{pmatrix} \in X \quad \text{and} \quad \xi_2 = \begin{pmatrix} \eta_2 \\ \zeta_2 \\ \phi_2 \\ \omega_2 \\ p_2 \\ q_2 \end{pmatrix} \in X,$$

the inner product in X is defined by the formula

$$\langle \xi_1, \xi_2 \rangle_X = \int_0^l (\eta_1''(x)\eta_2''(x) + \zeta_1(x)\zeta_2(x)) dx + \phi_1\phi_2 + \omega_1\omega_2 + p_1p_2 + q_1q_2.$$

It is easy to check that the norm $\|\xi\|_X = \sqrt{\langle \xi, \xi \rangle_X}$ is equivalent to the standard norm in $H^2(0, l) \times L_2(0, l) \times \mathbb{R}^4$ (see, e.g., [8, Ch. 3]), and hence, $(X, \|\cdot\|_X)$ is a Hilbert space.

In order to consider an abstract formulation of the boundary value problem (1)-(7), (10), let us introduce the linear operator $A : D(A) \rightarrow X$ and the element $B \in X$ as follows:

$$A : \xi = \begin{pmatrix} \eta \\ \zeta \\ \phi \\ \omega \\ p \\ q \end{pmatrix} \mapsto A\xi = \begin{pmatrix} \zeta \\ -\frac{1}{\rho}(c\eta'')'' + \gamma\phi \\ \omega \\ 0 \\ \gamma\phi + \frac{1}{m}(c\eta'')'|_{x=l} \\ -\frac{\varepsilon}{J}\eta''|_{x=l} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \psi \\ 0 \\ 1 \\ \psi(l) \\ \psi'(l) \end{pmatrix}, \quad (13)$$

where the domain of definition of A is

$$D(A) = \left\{ \xi \in X : \begin{array}{l} \eta \in H^4(0, l), \zeta \in H^2(0, l), \\ \zeta(0) = \zeta'(0) = 0, \\ p = \zeta(l), q = \zeta'(l) \end{array} \right\}, \quad (14)$$

functions $c(x) > 0$ and $\psi(x)$ are assumed to be of class $C^2[0, l]$; $J > 0$ and γ are constants.

Let $(y(x, t), \tilde{z}(x, t), \varphi_T(t), \tilde{\varphi}_R(t))$ be a classical solution of the boundary-value problem (1)-(7), (10) with controls $(u_T(t), u_R(t))$ for $0 \leq t < \tau$, $\tau \leq +\infty$. Defining

$$\xi_T(t) = \begin{pmatrix} y(\cdot, t) \\ \dot{y}(\cdot, t) \\ \varphi_T(t) \\ \dot{\varphi}_T(t) \\ \dot{y}(l, t) \\ \dot{y}'(l, t) \end{pmatrix}, \quad \xi_R(t) = \begin{pmatrix} \tilde{z}(\cdot, t) \\ \dot{\tilde{z}}(\cdot, t) \\ \tilde{\varphi}_R(t) \\ \dot{\tilde{\varphi}}_R(t) \\ \dot{\tilde{z}}(l, t) \\ \dot{\tilde{z}}'(l, t) \end{pmatrix}, \quad (15)$$

we see that $\xi_T(t) \in D(A)$ and $\xi_R(t) \in D(A)$ for all $t \in [0, \tau)$. Consider the pair (A_T, B_T) obtained from (A, B) by placing $\psi(x) = \psi_T(x)$, $c(x) = c_z(x)$, $J = J_3$, $\gamma = 0$ in (13). Similarly, let the pair (A_R, B_R) be obtained from (A, B) by plugging $\psi(x) = -x$, $c(x) = c_y(x)$, $J = J_2$, $\gamma = g \sin \varphi_R^0$. Then the boundary-value problem (1)-(7), (10) is reduced to the following control system:

$$\dot{\xi}_T = A_T \xi_T + B_T u_T, \quad (16)$$

$$\dot{\xi}_R = A_R \xi_R + B_R u_R, \quad (17)$$

where (ξ_T, ξ_R) is the state and (u_T, u_R) is the control. In the sequel, we treat this control system as an abstract formulation of (1)-(7), (10) with $\xi_T, \xi_R \in X$ and $u_T, u_R \in \mathbb{R}$. We see that (16), (17) is separated into two parts, therefore, the stabilization problem may be solved independently for ξ_T and ξ_R . The basic result we shall prove is the following

Theorem 1. *Consider the abstract Cauchy problem on $t \geq 0$:*

$$\dot{\xi}(t) = A\xi(t) + Bu, \quad (18)$$

$$\xi(0) = \xi_0 \in X, \quad (19)$$

where A, B are given by (13),

$$u = -\frac{1}{\beta} \left\{ k\omega + \left(\alpha - \gamma \left(\int_0^l \rho\psi \, dx + m\psi(l) \right) \right) \phi + \right. \\ \left. + \int_0^l c\eta''\psi'' \, dx + \left(c\eta''\psi' - (c\eta'')'\psi \right) \Big|_{x=0} - \gamma \left(\int_0^l \rho\eta \, dx + m\eta(l) \right) \right\}, \quad (20)$$

$\alpha > 0, \beta > 0$ are large enough constants, and $k > 0$ is arbitrary.

Then the Cauchy problem (18), (19) with (20) is well-posed on $t \geq 0$ (in the sense of mild solutions), and the feedback control (20) strongly stabilizes the equilibrium $\xi = 0$ of the control system (18), i.e., for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for every solution of (18)-(20),

$$\|\xi_0\|_X < \delta \Rightarrow \|\xi(t)\|_X < \varepsilon, \quad \forall t \geq 0.$$

Moreover, if a semitrajectory $\{\xi(t)\}_{t \geq 0}$ of (18), (20) is precompact in X then the set of its limit points (as $t \rightarrow +\infty$) is an invariant subset of $Z_0 = \{\xi \in X : \omega = 0\}$.

Proof. Consider a quadratic functional on X :

$$2V(\xi) = \alpha\phi^2 + \beta\omega^2 + \int_0^l \left\{ (\zeta - \psi\omega)^2 \rho + \eta''^2 c \right\} dx + \\ + m \{p - \psi(l)\omega\}^2 + J \{q - \psi'(l)\omega\}^2 - 2\gamma\phi \left\{ \int_0^l \eta\rho \, dx + m\eta(l) \right\}. \quad (21)$$

Let us compute the time-derivative of V along trajectories of (18) when $\xi \in D(A)$:

$$\dot{V}(\xi) = \langle \nabla V(\xi), A\xi + Bu \rangle = \\ = \int_0^l \left(c\zeta''\eta'' - \zeta \cdot (c\eta'')'' \right) dx + \left(p(c\eta'')' - qc\eta'' \right) \Big|_l + \\ + \left\{ \alpha\phi + \beta\omega + \int_0^l \left(\psi \cdot (c\eta'')'' - \rho\gamma(\phi\psi + \eta) \right) dx + \right. \\ \left. + \left(c\psi'\eta'' - \psi \cdot (c\eta'')' - m\gamma(\phi\psi + \eta) \right) \Big|_{x=l} \right\} \omega. \quad (22)$$

By performing integration by parts with regard for conditions (14), we get:

$$\int_0^l \zeta \cdot (c\eta'')'' dx = \zeta \cdot (c\eta'')' \Big|_{x=0}^l - \int_0^l \zeta' \cdot (c\eta'')' dx =$$

$$\begin{aligned}
&= \left(\zeta \cdot (c\eta'')' - \zeta' c\eta'' \right) \Big|_{x=l} + \int_0^l \zeta'' c\eta'' dx, \\
\int_0^l \psi \cdot (c\eta'')'' dx &= \psi \cdot (c\eta'')' \Big|_{x=0}^l - \int_0^l \psi' (c\eta'')' dx = \\
&= \left(\psi \cdot (c\eta'')' - \psi' c\eta'' \right) \Big|_{x=0}^l + \int_0^l \psi'' c\eta_T'' dx.
\end{aligned}$$

Let us substitute these formulae into (22) and use boundary conditions $p = \zeta(l)$, $q = \zeta'(l)$ from (14). As a result, the expression for \dot{V} takes the following form:

$$\begin{aligned}
\dot{V}(\xi) &= \left\{ \left(\alpha - \gamma \left(\int_0^l \rho \psi dx + m\psi(l) \right) \right) \phi + \beta u + \right. \\
&\quad \left. + \int_0^l c\psi'' \eta'' dx + \left(c\psi' \eta'' - \psi \cdot (c\eta'')' \right) \Big|_{x=0} - \gamma \left(\int_0^l \rho \eta dx + m\eta(l) \right) \right\} \omega. \quad (23)
\end{aligned}$$

If u is defined by (20) then formula (23) yields

$$\dot{V}(\xi) = -k\omega^2 \leq 0, \quad (k = \text{const} > 0). \quad (24)$$

The next step is to prove that $V(\xi)$ satisfies the following estimates

$$M_1 \|\xi\|_X^2 \leq 2V(\xi) \leq M_2 \|\xi\|_X^2 \quad (25)$$

with some constants $0 < M_1 \leq M_2 < +\infty$. On one hand, by exploiting inequalities $(a+b)^2 \leq 2a^2 + 2b^2$ and $2ab \leq a^2 + b^2$ in (21), we obtain

$$\begin{aligned}
2V(\xi) &\leq \alpha\phi^2 + \beta\omega^2 + \int_0^l \left(c\eta''^2 + 2(\zeta^2 + \psi^2\omega^2)\rho \right) dx + \\
&\quad + 2m \left(p^2 + \psi^2(l)\omega^2 \right) + 2J \left(q^2 + \psi'^2(l)\omega^2 \right) + \\
&\quad + \gamma^2\phi^2 + 2 \left(\int_0^l \eta\rho dx \right)^2 + 2(m\eta(l))^2. \quad (26)
\end{aligned}$$

Then the Cauchy-Schwartz inequality implies

$$\left(\int_0^l \eta\rho dx \right)^2 \leq \int_0^l \eta^2 dx \int_0^l \rho^2 dx, \quad (27)$$

$$\eta^2(l) = \left(\int_0^l \eta' dx \right)^2 \leq \int_0^l dx \int_0^l \eta'^2 dx. \quad (28)$$

The functions $\eta(x)$ and $\eta'(x)$ subject to the boundary conditions $\eta(0) = \eta'(0) = 0$ satisfy Friedrichs' inequalities of the following form (cf. [4, p. 440]):

$$\int_0^l \eta^2 dx \leq \frac{l^2}{2} \int_0^l \eta'^2 dx \leq \frac{l^4}{4} \int_0^l \eta''^2 dx. \quad (29)$$

By using inequalities (27)-(29) we conclude that

$$\begin{aligned} & \left(\int_0^l \eta \rho dx \right)^2 + m^2 \eta^2(l) \leq \int_0^l \eta^2 dx \int_0^l \rho^2 dx + \\ & + l m^2 \int_0^l \eta'^2 dx \leq \frac{l^3}{2} \left(m^2 + \frac{l}{2} \int_0^l \rho^2 dx \right) \int_0^l \eta''^2 dx. \end{aligned} \quad (30)$$

Application of this inequality in (26) yields an estimate $2V(\xi) \leq M_2 \|\xi_T\|_X^2$,

$$\begin{aligned} M_2 = \max & \left\{ \alpha + \gamma^2, 2m, 2J, 2 \max_{x \in [0, l]} \rho(x), \right. \\ & \beta + 2 \int_0^l \psi^2 \rho dx + 2J\psi'^2(l) + 2m\psi^2(l), \\ & \left. l^3 \left(m^2 + \frac{l}{2} \int_0^l \rho^2 dx \right) + \max_{x \in [0, l]} c(x) \right\}. \end{aligned}$$

On the other hand, we see that the inequality $a^2 = (a-b+b)^2 \leq 2(a-b)^2 + 2b^2$ implies $(a-b)^2 \geq a^2/2 - b^2$. By using the latter together with $-2ab \geq -\varkappa^2 a^2 - b^2/\varkappa^2$ ($\varkappa \neq 0$) in (21), we get:

$$\begin{aligned} 2V(\xi) & \geq \alpha \phi^2 + \beta \omega^2 + \int_0^l \left(c \eta''^2 + \frac{\rho}{2} \zeta^2 - \rho \psi^2 \omega^2 \right) dx + \\ & + m \left(\frac{p^2}{2} - \psi^2(l) \omega^2 \right) + J \left(\frac{q^2}{2} - \psi'^2(l) \omega^2 \right) - \\ & - \varkappa^2 \gamma^2 \phi^2 - \frac{1}{\varkappa^2} \left(\int_0^l \eta \rho dx + m \eta(l) \right)^2 \geq \\ & \geq (\alpha - \varkappa^2 \gamma^2) \phi^2 + \frac{m}{2} p^2 + \frac{J}{2} q^2 + \frac{1}{2} \int_0^l \zeta^2 \rho dx + \end{aligned}$$

$$\begin{aligned}
& + \left(\beta - \int_0^l \rho \psi^2 dx - m \psi^2(l) - J \psi'^2(l) \right) \omega^2 + \\
& + \left\{ \min_{[0,l]} c - \frac{l^3}{\varkappa^2} \left(m^2 + \frac{l}{2} \int_0^l \rho^2 dx \right) \right\} \int_0^l \eta''^2 dx. \tag{31}
\end{aligned}$$

We have also used the inequality (30) here. From (31) we conclude that $2V(\xi) \geq M_1 \|\xi\|_X^2$ and

$$M_1 = \min \left\{ \alpha - \varkappa^2 \gamma^2, \frac{m}{2}, \frac{J}{2}, \frac{1}{2} \min_{x \in [0,l]} \rho(x), \right.$$

$$\left. \beta - \int_0^l \rho \psi^2 dx - m \psi^2(l) - J \psi'^2(l), \right.$$

$$\left. \min_{x \in [0,l]} c(x) - \frac{l^3}{\varkappa^2} \left(m^2 + \frac{l}{2} \int_0^l \rho^2 dx \right) \right\} > 0$$

provided that

$$\varkappa^2 > \frac{l^3}{\min_{x \in [0,l]} c(x)} \left(m^2 + \frac{l}{2} \int_0^l \rho^2 dx \right),$$

$$\alpha > \varkappa^2 \gamma^2, \quad \beta > \int_0^l \rho \psi^2 dx + m \psi^2(l) + J \psi'^2(l).$$

For the rest of this paper, we assume that constants α , β , and \varkappa satisfy the above inequalities.

The estimate (25) shows that the two norms $\|\xi\|_X$ and $\|\xi\|_V = \sqrt{V(\xi)}$ are equivalent in X . Let us write the closed-loop system (18) with the control u defined by (20) as $\dot{\xi} = \tilde{A}\xi$, where $D(\tilde{A}) = D(A)$ is dense in X . From inequality (24) it follows that the operator \tilde{A} is dissipative in X equipped with the norm $\|\cdot\|_V$. Then the Lumer-Phillips theorem [9, Chap. 1.4] implies that \tilde{A} is the infinitesimal generator of a C_0 semigroup of contractions, $\{e^{t\tilde{A}}\}_{t \geq 0}$, on X (with respect to the norm $\|\cdot\|_V$). It means that the Cauchy problem (18)-(20) has the unique mild solution $\xi(t) = e^{t\tilde{A}}\xi_0$, $t \geq 0$, for every $\xi_0 \in X$, and the above solution is classical if $\xi_0 \in D(A)$. As $\{e^{t\tilde{A}}\}_{t \geq 0}$ is contractive (under an equivalent renormalization in X), then

$$\|\xi(t)\|_V \leq \|\xi_0\|_V, \quad \forall t \geq 0.$$

This implies, taking into account the estimate (25), that

$$\|\xi(t)\|_X^2 \leq \frac{2V(\xi(t))}{M_1} \leq \frac{2V(\xi_0)}{M_1} \leq \frac{M_2}{M_1} \|\xi_0\|_X^2.$$

The above inequality proves strong stability of the equilibrium $\xi = 0$ in the sense of Lyapunov (we may choose $\delta(\varepsilon) = \varepsilon \sqrt{M_1/M_2}$ in the definition of stability).

To conclude the proof we apply LaSalle's invariance principle [6, 10] (cf. [14, Lemma 2]) with the functional $V(\xi)$: if a semitrajectory $\{\xi(t)\}_{t \geq 0}$ is precompact then its ω -limit set, $\Omega(\xi_0)$, is a non-empty and semi-invariant subset of $\{\xi \in D(A) : \dot{V}(\xi) = 0\} = Z_0$. \square

Remark. By applying the formula (20) to control systems (16), (17) separately and using the representation (15), one can write the feedback control proposed as follows:

$$\begin{aligned} u_T &= -\frac{1}{\beta} \left\{ \alpha \varphi_T + k \dot{\varphi}_T + \int_0^l c_z y'' \psi'' dx + \left(c_z y'' \psi' - (c_z y'')' \psi \right) \Big|_{x=0} \right\}, \\ u_R &= -\frac{1}{\beta} \left\{ \alpha \tilde{\varphi}_R + k \dot{\tilde{\varphi}}_R - c_y \tilde{z}'' \Big|_{x=0} + \right. \\ &\quad \left. + g \left(\int_0^l (x \tilde{\varphi}_R - \tilde{z}) \rho dx + m(l \tilde{\varphi}_R - \tilde{z} \Big|_{x=l}) \right) \sin \varphi_R^0 \right\}. \end{aligned}$$

To implement these controls in practice, it is sufficient to compute u_T and u_R depending on the measurements of φ_T , $\tilde{\varphi}_R$, $\dot{\varphi}_T$, $\dot{\tilde{\varphi}}_R$, y , \tilde{z} at each $t \geq 0$, and then apply formulae (11), (12) to find torques M_T and M_R . An advantage of this approach is that no information about the time-derivatives of $y(x, t)$ and $\tilde{z}(x, t)$ is needed.

3 Conclusions

A feedback control has been derived to stabilize the equilibrium of a differential equation in a Hilbert space that describes the motion of a flexible beam with a tip mass. Although the main result of this paper concerns non-asymptotic stability, further analysis of the limit behavior of controlled trajectories is possible by means of the invariance principle. The main difficulty in this direction is to prove that the semitrajectories are precompact,

which is not an easy task in general (see, e.g., [2, 15]). We do not study the compactness issue here, leaving it for future work.

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